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# AN ESTIMATE OF THE ISOVARIANT BORSUK-ULAM CONSTANT

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ABSTRACT. We shall discuss the isovariant Borsuk-Ulam constant determined from the weak isovariant Borsuk-Ulam theorem. We first illustrate some properties of the Borsuk-Ulam constant and next provide an estimate of the isovariant Borsuk-Ulam constant for the special unitary group $\mathrm{SU}(n)$.

## 1. BACKGROUND

Borsuk-Ulam type results for $G$-maps between (linear) $G$-spheres were studied by many researchers and various generalizations were shown. In particular, the following generalization is well known; see [3] for example.

**Theorem 1.1.** *Let $G$ be $(C_p)^k$ a product of cyclic groups of prime order $p$ or $T^k$ a ($k$-dimensional) torus. Suppose that $G$ acts smoothly and fixed-point-freely on spheres $S_1$ and $S_2$. If there exists a (continuous) $G$-map $f : S_1 \to S_2$. then the inequality*

$$\dim S_1 \leq \dim S_2$$

*holds.*

On the other hand, T. Bartsch [1] proved that such a Borsuk-Ulam result does not hold for $G$ not being a $p$-toral group. A compact Lie group $G$ is called *$p$-toral* if there is an exact sequence $1 \to T \to G \to P \to 1$, where $T$ is a torus and $P$ is a finite $p$-group.

As a variation of the Borsuk-Ulam theorem, the isovariant Borsuk-Ulam theorem was first studied by A. G. Wasserman [9]. Let $G$ be a compact Lie group. A $G$-map $f : X \to Y$ is called *$G$-isovariant* if $f$ preserves the isotropy subgroups, i.e., $G_x = G_{f(x)}$ for any $x \in X$. In other words, it is a $G$-map such that $f_{|G(x)} : G(x) \to Y$ is injective on each orbit $G(x)$ of $x \in X$. From Wasserman's results, one sees the following.

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**Theorem 1.2** (Isovariant Borsuk-Ulam theorem). *Let $G$ be a solvable compact Lie group. If there exists a $G$-isovariant map $f : SV \to SW$ between linear $G$-spheres, then*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

*holds.*

Wasserman conjectures that this theorem holds for all finite groups. This is unsolved at present; however, we showed a *weak version* of the isovariant Borsuk-Ulam theorem for an *arbitrary* compact Lie group.

**Theorem 1.3** (Weak isovariant Borsuk-Ulam theorem ([5, 6])). *There exists a positive constant $c > 0$ such that*

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

*for any pair of representations $V$ and $W$ with a $G$-isovariant map $f : SV \to SW$.*

*Definition.* The *isovariant Borsuk-Ulam constant* $c_G$ of $G$ is defined to be the supremum of such a constant $c$. (If $G = 1$, then set $c_G = 1$ as convention.)

When $c_G = 1$, $G$ is called a *Borsuk-Ulam group* (BUG for short); namely, a Borsuk-Ulam group $G$ is a compact Lie group for which the isovariant Borsuk-Ulam theorem holds. In particular, a solvable compact Lie group is a Borsuk-Ulam group by Theorem 1.2, and several nonsolvable Borsuk-Ulam finite groups are also known; for the detail, see [7, 8, 9]. However, no one knows connected Borsuk-Ulam groups other than a torus. Therefore we would like to investigate $c_G$ and provide some estimates at least. We illustrate general properties of $c_G$ in section 2 and we provide an estimate $c_G$ for $G = \mathrm{U}(n)$ in section 3; in fact, we notice

$$c_{\mathrm{U}(n)} \geq \frac{n}{n+1}$$

whose complete proof will be written elsewhere.

## 2. Properties of $c_G$

The following result is a generalization of Wasserman's result and is proved by a similar argument as in [9].

**Proposition 2.1.** *If $1 \to K \to G \to Q \to 1$ is an exact sequence of compact Lie groups, then*

$$\min\{c_K, c_Q\} \leq c_G \leq c_Q.$$

*In particular, if $K$ is a Borsuk-Ulam group, then $c_G = c_Q$.*

Using this inductively, we have

**Corollary 2.2.** *If* $1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$, *then*

$$\min_{1\le i\le r}\{c_{H_i/H_{i-1}}\} \le c_G.$$

As an example, one sees the following.

**Example 2.3.** *It follows that* $c_{\mathrm{U}(n)} = c_{\mathrm{SU}(n)} = c_{\mathrm{PSU}(n)}$. *In particular,* $c_{\mathrm{SU}(2)} = c_{\mathrm{SO}(3)}$ *since* $\mathrm{PSU}(2) \cong \mathrm{SO}(3)$.

*Proof.* There is an exact sequence

$$1 \to C_n \to S^1 \times \mathrm{SU}(n) \to \mathrm{U}(n) \to 1.$$

Since $C_n$ is a Borsuk-Ulam group, it follows from Proposition 2.1 that $c_{\mathrm{U}(n)} = c_{S^1\times \mathrm{SU}(n)}$. Next, there is an exact sequence

$$1 \to S^1 \to S^1 \times \mathrm{SU}(n) \to \mathrm{SU}(n) \to 1.$$

Since $S^1$ is a Borsuk-Ulam group, it follows that $c_{S^1\times\mathrm{SU}(n)} = c_{\mathrm{SU}(n)}$. Thus $c_{\mathrm{U}(n)} = c_{\mathrm{SU}(n)}$. Since the center of $\mathrm{SU}(n)$ is isomorphic to $C_n$, it follows that $c_{\mathrm{PSU}(n)} = c_{\mathrm{SU}(n)}$. □

## 3. Estimation of $c_{\mathrm{U}(n)}$

Let $T$ denote the maximal torus $T$ of $\mathrm{U}(n)$ given by diagonal matrices:

$$T = \left\{ \begin{pmatrix} t_1 & & O \\ & \ddots & \\ O & & t_n \end{pmatrix} \,\middle|\, t_i \in S^1\,(\subset \mathbb{C}) \right\}.$$

We set

$$d_{\mathrm{U}(n)} = \sup\left\{ \frac{\dim U^T}{\dim U} \,\middle|\, U : \text{nontrivial irreducible } \mathrm{U}(n)\text{-representation} \right\}.$$

In order to estimate $c_{\mathrm{U}(n)}$, we use the fact $c_{\mathrm{U}(n)} \ge 1 - d_{\mathrm{U}(n)}$ deduced from a result of [6].

**Theorem 3.1.** $d_{\mathrm{U}(n)} = \dfrac{1}{n+1}$, *and hence* $c_{\mathrm{U}(n)} \ge \dfrac{n}{n+1}$.

This is proved by representation theory. The irreducible complex representations of $\mathrm{U}(n)$ are parametrized by $\lambda$ in

$$\Lambda = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \ge \cdots \ge \lambda_n\}.$$

Let $V_\lambda$ denote the irreducible $\mathrm{U}(n)$-representation corresponding to $\lambda \in \Lambda$. (Then $\lambda$ is the highest weight of $V_\lambda$.) Since $\mathrm{Res}_T : R(\mathrm{U}(n)) \to R(T)^{W_n}$ is isomorphic, where $W_n \cong S_n$ is the Weyl group of $\mathrm{U}(n)$, the character $\chi_\lambda$ of $\mathrm{Res}_T V_\lambda$ is a homogenous symmetric Laurent polynomial in $\mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ with a form

$$\chi_\lambda(t) = \sum_{\mu \in \mathbb{Z}^n} m_\lambda(\mu) t^\mu = \sum_{\mu \in \mathbb{Z}^n} m_\lambda(\mu) t_1^{\mu_1} \cdots t_n^{\mu_n} \quad (t = \mathrm{diag}(t_1, \cdots, t_n) \in T).$$

The coefficient $m_\lambda(\mu)$ is the multiplicity of a weight $\mu$, i.e., the dimension of the weight space corresponding to $\mu$:

$$m_\lambda(\mu) = \dim\{v \in V_\lambda \,|\, t \cdot v = t^\mu v \text{ for all } t \in T\} \geq 0.$$

Let $M_\lambda := \{\mu \in \mathbb{Z}^n \,|\, |\mu| = |\lambda| \text{ and } \mu \preceq \lambda\}$, which is a finite set. Here $|\mu| = \sum_{i=1}^n \mu_i$, and $\preceq$ is the dominant order on $\mathbb{Z}^n$ defined by

$$\mu \preceq \lambda \iff \sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \lambda_i \ (1 \leq \forall k \leq n).$$

The following results can be found in [2, 4].

**Proposition 3.2.** *Let $\lambda \in \Lambda$ and $\mu \in \mathbb{Z}^n$.*

(1) $m_\lambda(\mu) \neq 0 \iff \mu \in M_\lambda$.
(2) $m_\lambda(\lambda) = 1$ *for* $\lambda \in \Lambda$.
(3) $m_\lambda(w \cdot \mu) = m_\lambda(\mu)$ *for any* $w \in W_n$, *where* $w \cdot \mu = (\mu_{w^{-1}(1)}, \cdots, \mu_{w^{-1}(n)})$.
(4) $W_n$ *acts on* $M_\lambda$ *by permutation as in* (3) *and for any* $\mu \in M_\lambda$, $W_n(\mu) \cap \Lambda$ *consists of one element. Therefore* $M_\lambda \cap \Lambda$ *is a complete system of representatives of* $M_\lambda / W_n$.

Thus the character has a form

$$\chi_\lambda(t) = \sum_{\mu \in M_\lambda} m_\lambda(\mu) t^\mu = \sum_{\mu \in M_\lambda \cap \Lambda} m_\lambda(\mu) P_\mu(t),$$

where $P_\mu(t) = \sum_{\nu \in W_n(\mu)} t^\nu$.

**Proposition 3.3.** *Let $G = \mathrm{U}(n)$ and $\lambda \in \Lambda$.*

(1) $\dim V_\lambda = \chi_\lambda(1) = \sum_{\mu \in M_\lambda} m_\lambda(\mu)$.
(2) $\dim V_\lambda^T = m_\lambda(0)$, *the constant term of* $\chi_\lambda(t)$.
(3) $\dim V_\lambda^T > 0 \iff 0 \in M_\lambda \iff \lambda \in \Lambda_0 := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \,|\, \lambda_1 \geq \cdots \geq \lambda_n,\ \sum_i \lambda_i = 0\}$.

Furthermore, the dimension of $V_\lambda$ is described in terms of the highest weight $\lambda \in \Lambda$.

**Proposition 3.4** (dimension formula for $U(n)$ ([2, 4])).

$$\dim V_\lambda = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{i<j}(j - i)}.$$

On the other hand, computation of the multiplicity is not so easy (if $\lambda$ is large); however several multiplicity formulas are known; for example, Freudenthal formula, Kostant formula, and combinatorially $m_\lambda(\mu)$ can be given as a Kostka number (= the number of certain semi-standard Young tableaux). We use Freudenthal's multiplicity formula; see [4] for example.

3.1. **Outline of proof of Theorem 3.1.** We may assume $\lambda \in \Lambda_0$ and $\lambda \neq 0$, since $\dim V_\lambda^T = 0$ if $\lambda \notin \Lambda_0$. Let $(-,-)$ denote the (standard) inner product on $\mathbb{R}^n$. Let $\alpha_{ij} = e_i - e_j$ for $i \neq j$, where $e_i$ is the $i$-th fundamental unit vector. All $\alpha_{ij}$ form the root system of type $A_{n-1}$. Let $R_+ = \{\alpha_{ij} \mid 1 \le i < j \le n\}$ the set of positive roots and set

$$\rho := \frac{1}{2}\sum_{\alpha\in R_+} \alpha = \left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-3}{2}, -\frac{n-1}{2}\right).$$

Applying Freudenthal's multiplicity formula to $\mu = 0$, we have an inequality

$$(*) :\ m_\lambda(0)K_\lambda \le 2n(n-1)d\sum_{k=1}^{d} m_\lambda(\mu_k),$$

where $K_\lambda := \|\lambda\|^2 + 2(\lambda,\rho)$ and $\mu_k := k\alpha_{1n} = (k, 0, \cdots, 0, -k) \in \Lambda_0$. Since $\mu_k \in M_\lambda$ $(1 \le k \le d)$, $\chi_\lambda(t)$ has a form

$$\chi_\lambda(t) = m_\lambda(0) + \sum_{k=1}^{d} m_\lambda(\mu_k)P_{\mu_k}(t) + \text{ other terms},$$

where $P_{\mu_k}(t) = \sum_{i\neq j} t^{k\alpha_{ij}} = \sum_{i\neq j} t_i^k t_j^{-k}$, which has $n(n-1)$ terms. This shows

$$\dim V_\lambda = \chi_\lambda(1) \ge m_\lambda(0) + \sum_{k=1}^{d} m_\lambda(\mu_k)n(n-1).$$

Using the inequality $(*)$, we obtain

$$\dim V_\lambda \ge \left(1 + \frac{K_\lambda}{2d}\right) m_\lambda(0).$$

Since $K_\lambda = \|\lambda\|^2 + 2(\lambda,\rho) \ge \lambda_1^2 + \lambda_n^2 + (n-1)(\lambda_1 - \lambda_n)$, it follows that

$$\frac{\dim V_\lambda^T}{\dim V_\lambda} \le \frac{1}{n+1}.$$

On the other hand, applying the multiplicity formula to $\lambda = \mu_1$, one sees

$$\dim V_{\mu_1}^T = n - 1,$$

and by the dimension formula, $\dim V_{\mu_1} = (n+1)(n-1)$. Hence it follows that

$$\frac{\dim V_{\mu_1}^T}{\dim V_{\mu_1}} = \frac{1}{n+1}.$$

Thus we have $d_{\mathrm{U}(n)} = \dfrac{1}{n+1}$. $\square$

*Remark.* In case of $n = 2$, the theorem provide an estimate $c_{U(2)} \geq 2/3$; however, this may be improved by a further argument; in fact, we show that $c_{U(2)} \geq 4/5$ in [6].

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